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# Moyal-like form of the star product for generalized $SU(2)$ Stratonovich–Weyl symbols

**A B Klimov and P Espinoza**[1]

Departamento de Física, Universidad de Guadalajara, Revolución 1500, 44410, Guadalajara, Jal., México

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**Abstract**

The star product is the basic tool used in the phase-space formulation of quantum mechanics. We find a differential form of the star product for a class of $s$-parametrized $SU(2)$ Stratonovich–Weyl symbols which appear in the phase-space representation of spin-like systems. The limit of large spin is considered and the asymptotic form of the differential operator defining the star product is obtained.

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## 1. Introduction

Since the seminal paper of Wigner [1], the phase-space methods have been successfully applied in different branches of quantum mechanics (see, for example, [2, 3]). The phase-space representation allows us to reformulate standard quantum mechanics on the classical language of phase spaces and functions defined on them [4], providing a very useful insight into quantum–classical correspondence in non-relativistic quantum mechanics. According to the Moyal's formulation of quantum mechanics, both states and observables are considered as functions on a given phase space, in such a way that average values are computed as in classical statistical mechanics: by integrating over the phase space of some quasi-distribution function with the Weyl symbol of a corresponding operator. The axiomatic approach to the phase-space formulation of quantum mechanics was developed by Stratonovich [5]. Nowadays this approach is known as 'the Stratonovich–Weyl correspondence'. According to this approach, we associate each operator $\hat{f}$ with its symbol $f(\Omega)$, a $c$-number function defined in the corresponding phase space. The cornerstone of the Stratonovich–Weyl correspondence is a specific symbol calculus, the so-called star (or twisted) product, which associates the product of two operators $\hat{f}_1\hat{f}_2$ with an (associative) star product $f_1(\Omega) * f_2(\Omega)$. This star product allows us to replace the standard manipulations with operators in the Hilbert space by a differential (or integral) operator acting on the product of Weyl's symbols. On the other hand, the introduction of an associative but non-commutative star product to the algebra of classical

[1] Address for correspondence: CUALTOS, Enrique Diaz de Leon, S/N, Lagos de Moreno, Jal., Mexico.

observables leads to a specific quantization procedure (quantization by deformation) [6], now applied in M-physics [7]. Obviously, the (invertible) map $\hat{f} \rightarrow f(\Omega)$ (and, correspondingly, the form of the star product) depends on the ordering rules of functions of noncommutative operators. This can be taken into account by introducing an additional index, $s$, which specifies a certain operator ordering, such that $\hat{f} \rightarrow f^{(s)}(\Omega)$.

The star product can be represented in differential and integral forms. Although a general expression for the integral representation of the star product is easy to obtain, it is not very useful for performing calculations (except the simplest case of the Heisenberg–Weyl group). In this sense the differential form of the star product is more convenient. Both integral and differential representations of the star product for the Heisenberg–Weyl group have been extensively explored since Moyal's paper [4]. The case of the $SU(2)$ group has not been explored to the same extent. In this paper we find an exact differential form of the star product for the spin-like systems and discuss the quasiclassical limit for large spin. In section 2 we give a short introduction to the principal mathematical ideas of this paper. In section 3 we briefly discuss the Stratonovich–Weyl mapping for the spin-like systems. In section 4 we obtain the differential form of the star product for the family of $s$-parametrized Stratonovich–Weyl symbols and discuss the result, giving an example in section 5. In section 6 the limit for large spin is considered and the asymptotic form of the differential operator defining the star product is obtained.

## 2. A mathematical overview

In the phase-space formulation of quantum mechanics in the flat $q$–$p(\alpha$–$\alpha^*)$ space a family of $s$-parametrized quasi-distribution functions $W_\rho^{(s)}(\alpha)$ (related to different ordering of the position and momentum operators) naturally appears [8, 9]. These quasiprobability distributions are built as mean values of the operator kernel which is the Fourier transform of the (ordered) displacement operator from the Heisenberg–Weyl group representation [9],

$$W_\rho^{(s)}(\alpha) = \mathrm{Tr}(\hat{w}_s(\alpha)\rho) \tag{1}$$

$$\hat{w}_s(\alpha) = \frac{1}{\pi}\int \mathrm{d}^2\xi\, D(\xi)\exp[\xi^*\alpha - \xi\alpha^* + s|\xi|^2/2], \tag{2}$$

where

$$D(\xi) = \exp(\xi a^\dagger - \xi^* a),$$

$\alpha = (q + \mathrm{i}p)/\sqrt{2}$ ($\hbar = 1$) and $\rho$ is the system density matrix. The values $0, +1, -1$ of the parameter $s$ correspond to the Wigner $W$-function, the Glauber–Sudarshan $P$ function and the Husimi $Q$ function. In the same way as (1) the $s$-parametrized Weyl symbol of an arbitrary operator $\hat{f}$ is introduced:

$$W_f^{(s)}(\alpha) = \mathrm{Tr}(\hat{w}_s(\alpha)\hat{f}).$$

The corresponding star product is defined as [4, 10, 11],

$$W_f^{(s)}(\alpha) * W_g^{(s)}(\alpha) = \exp\left[-\frac{s}{2}\left(\partial_\alpha^{(f)}\partial_{\alpha^*}^{(g)} + \partial_{\alpha^*}^{(f)}\partial_\alpha^{(g)}\right)\right] \times \exp\left[\tfrac{1}{2}\left(\partial_{\alpha^*}^{(f)}\partial_\alpha^{(g)} - \partial_\alpha^{(f)}\partial_{\alpha^*}^{(g)}\right)\right] W_f^{(s)}(\alpha)W_g^{(s)}(\alpha). \tag{3}$$

In addition to the Heisenberg–Weyl case, the Moyal quantization programme has been realized in the spin-like systems possessing $SU(2)$ group symmetry. The phase-space description of spin systems was initiated by Stratonovich [5], Beresin [12] and Agarwal [13] (see also [14–19], where different types of the quasiprobability distribution function on the sphere $(\theta, \phi) \in \mathcal{S}_2$

have been discussed). Recently, the Moyal quantization scheme has been generalized for quantum systems possessing a (connected and finite-dimensional) group of dynamical symmetry [20] and a simple algorithm for constructing the ($s$-parametrized) Stratonovich–Weyl such as kernels $\hat{w}_s(\Omega)$ (as a function defined on the 'classical' phase-space $X$, $\Omega \in X$) was proposed. The general rules to associate to each operator $\hat{f}$ acting on a Hilbert space, a function $W_f^{(s)}$ ($s$-parametrized symbol of $\hat{f}$) defined on the phase space, are given by the 'Stratonovich–Weyl correspondence' [5, 17, 20]

$$W_f^{(s)}(\Omega) = \mathrm{Tr}(\hat{w}_s(\Omega)\hat{f}). \tag{4}$$

The star product of two Stratonovich–Weyl symbols is determined by the condition

$$W_f^{(s_1)} * W_g^{(s_2)} = W_{fg}^{(s)} \tag{5}$$

for any two operators $\hat{f}$, $\hat{g}$. From the Stratonovich–Weyl postulates and the above definition the following expression (integral representation) for the star product is obtained [17, 20]:

$$W_{fg}^{(s)} = \mathrm{Tr}(\hat{w}_s(\Omega)\hat{f}\hat{g}) = \int_X \int_X K^{(s,s_1,s_2)}(\Omega, \Omega_1, \Omega_2) W_f^{(s)}(\Omega_1) W_g^{(s)}(\Omega_2)\, \mathrm{d}\mu(\Omega_1)\, \mathrm{d}\mu(\Omega_2) \tag{6}$$

where $\mathrm{d}\mu(\Omega)$ is the invariant measure on $X$ and $K^{(s,s_1,s_2)}(\Omega, \Omega_1, \Omega_2)$ is the kernel defined as

$$K^{(s,s_1,s_2)}(\Omega, \Omega_1, \Omega_2) = \mathrm{Tr}[\hat{w}_s(\Omega)\hat{w}_{s_1}(\Omega_1)\hat{w}_{s_2}(\Omega_2)]. \tag{7}$$

The kernel (7) has, in general, a quite complicated form. In the case of the Heisenberg–Weyl group a differential form of the star product (6) was obtained by Moyal for the case of the Wigner mapping, and later generalized to an arbitrary ordering [10, 11]. The differential form of the star product allows us to introduce the so-called Moyal brackets

$$\{W_f^{(s_1)}, W_g^{(s_2)}\}_M = W_f^{(s_1)} * W_g^{(s_2)} - W_g^{(s_2)} * W_f^{(s_1)},$$

and thus, write down a differential evolution equation for quasi-distribution functions,

$$\mathrm{i}\partial_t W_\rho^{(s)} = \{W_H^{(s_1)}, W_\rho^{(s_2)}\}_M, \tag{8}$$

where $H$ is the system Hamiltonian.

In the classical limit, $\hbar \to 0$, the Moyal brackets (8) in the flat-space case turns to the Poisson brackets (on the $p$–$q$ phase space) according to

$$\frac{1}{\mathrm{i}\hbar}\{W_H^{(s_1)}, W_\rho^{(s_2)}\}_M = \{W_H^{(s_1)}, W_\rho^{(s_2)}\}_P + \mathrm{O}(\hbar).$$

A similar situation can be expected from the Moyal brackets for spin-like systems, where the inverse dimension of representation plays the role of the expansion parameter.

It is worth noting that structures similar to (6) and (7) naturally arise [21] in the tomographic representation of quantum mechanics [22].

## 3. The Stratonovich–Weyl correspondence for spin systems

The $s$-parametrized Stratonovich–Weyl kernel $\hat{w}_s(\theta, \phi)$ for systems possessing the $SU(2)$ dynamical symmetry group is introduced according to [5, 17, 20, 23]

$$\hat{w}_s(\theta, \phi) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} (C_{SS,L0}^{SS})^{-s} Y_{LM}^*(\theta, \phi) \hat{T}_{LM}^{(S)} = \hat{w}_s^\dagger(\theta, \phi), \tag{9}$$

where $Y_{LM}(\theta, \phi)$ are the spherical harmonics and $\hat{T}_{LM}^{(S)}$ are the irreducible tensor operators [24] which form an orthogonal operator basis in the space of $(2S+1) \times (2S+1)$ matrices and are defined as

$$\hat{T}_{LM}^{(S)} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-S}^{S} C_{Sm,LM}^{Sm'} |S, m'\rangle\langle S, m|. \tag{10}$$

Here $C^{Sm'}_{Sm,LM}$ are the Clebsch–Gordan coefficients which couple two representations of spin $S$ and $L$ $(0 \leqslant L \leqslant 2S)$ to a total spin $S$. The kernel $\hat{w}_s(\theta, \phi)$ is normalized according to

$$\operatorname{Tr} \hat{w}_s(\theta, \phi) = 1, \qquad \frac{2S+1}{4\pi} \int_{\mathcal{S}_2} \mathrm{d}\mu(\Omega)\, \hat{w}_s(\theta, \phi) = I, \tag{11}$$

where

$$\mathrm{d}\mu(\Omega) = \frac{2S+1}{4\pi}\, \mathrm{d}\Omega = \frac{2S+1}{4\pi} \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$$

is the invariant measure on the sphere.

The $s$-parametrized family of $SU(2)$ quasi-distribution functions (see [17] for review) are defined as follows:

$$W^{(s)}_{\rho}(\theta, \phi) = \operatorname{Tr}(\rho \hat{w}_s(\theta, \phi)), \tag{12}$$

where $\rho$ is the system density matrix. As well as in the Heisenberg–Weyl case, the value $s = 0$ corresponds to the Stratonovich–Weyl function, meanwhile $s = \pm 1$ leads to the Beresin contravariant $P$-symbol and covariant $Q$-symbol correspondingly.

The ($s$-parametrized) Stratonovich–Weyl symbols of the operator $\hat{f}$ (4)

$$W^{(s)}_{f}(\theta, \phi) = \operatorname{Tr}(\hat{f} \hat{w}_s(\theta, \phi)), \tag{13}$$

are covariant under rotations and provide the overlap relation

$$\frac{2S+1}{4\pi} \int_{\mathcal{S}_2} \mathrm{d}\Omega\, W^{(s)}_{g}(\theta, \phi) W^{(-s)}_{f}(\theta, \phi) = \operatorname{Tr}(\hat{g}\hat{f}). \tag{14}$$

The operator $\hat{f}$ can be reconstructed from its symbol $W^{(s)}_{f}(\theta, \phi)$ (13) through the following relation:

$$\hat{f} = \frac{2S+1}{4\pi} \int_{\mathcal{S}_2} \mathrm{d}\Omega\, \hat{w}_{-s}(\theta, \phi) W^{(s)}_{f}(\theta, \phi). \tag{15}$$

The kernel (7) defining the star product (6) takes the form

$$K^{(s,s_1,s_2)}(\theta, \phi; \theta_1, \phi_1; \theta_2, \phi_2) = \left(\frac{2S+1}{4\pi}\right)^2 \operatorname{Tr}[\hat{w}_s(\theta, \phi) \hat{w}_{s_1}(\theta_1, \phi_1) \hat{w}_{s_2}(\theta_2, \phi_2)]. \tag{16}$$

Unfortunately, this kernel has quite a complicated form and, thus, is not convenient for practical use.

## 4. A differential form for the star product

To find the differential form of the star product (5), i.e. to define a differential operator $\hat{L}_{fg}(\theta, \phi)$, such that

$$W^{(s)}_{fg} = W^{(s_1)}_{f} * W^{(s_2)}_{g} = \hat{L}^{(s)}_{fg}(\theta, \phi)[W^{(s_1)}_{f} W^{(s_2)}_{g}], \tag{17}$$

we make use of the reconstruction relation (15) for the product $\hat{f}\hat{g}$,

$$\hat{f}\hat{g} = \frac{2S+1}{4\pi} \int_{\mathcal{S}_2} \mathrm{d}\Omega\, \hat{w}_{-s}(\theta, \phi) W^{(s)}_{fg}(\theta, \phi), \tag{18}$$

and then express $W^{(s)}_{fg}(\theta, \phi)$ in the form (17). (It worth noting that the explicit expression (16) is not very suitable for calculations.)

Let us consider two operators $\hat{f}$ and $\hat{g}$ from the $2S+1$-dimensional representation of the universal enveloping algebra of $su(2)$. We represent both operators as series on the irreducible tensor operators $\hat{T}_{lk}^{(S)}$ (10)

$$\hat{f} = \sum_{l=0}^{2S}\sum_{k=-l}^{l} f_{lk}\hat{T}_{lk}^{(S)}, \qquad \hat{g} = \sum_{l=0}^{2S}\sum_{k=-l}^{l} g_{lk}\hat{T}_{lk}^{(S)}. \tag{19}$$

The degree of non-linearity (on the generators of the $su(2)$ algebra) of the operators (19), $\deg\hat{f}$, $\deg\hat{g}$ is defined by the maximum value of $l$, such that $f_{lk}\neq 0$, $g_{lk}\neq 0$. It is easy to observe that the ($s$-parametrized) Stratonovich–Weyl symbols of the operators $\hat{f}$ and $\hat{g}$ take the form

$$\begin{aligned} W_f^{(s)}(\theta,\phi) &= \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{l,k}\left(C_{SS,L0}^{SS}\right)^{-s} f_{lk}Y_{lk}(\theta,\phi), \\ W_g^{(s)}(\theta,\phi) &= \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{l,k}\left(C_{SS,L0}^{SS}\right)^{-s} g_{lk}Y_{lk}(\theta,\phi). \end{aligned} \tag{20}$$

For the product $\hat{f}\hat{g}$ we have

$$\hat{f}\hat{g} = \sum_{l_1=0}^{2S}\sum_{k_1=-l_1}^{l_1}\sum_{l_2=0}^{2S}\sum_{k_2=-l_2}^{l_2} f_{l_1k_1}g_{l_2k_2}\hat{T}_{l_1k_1}^{(S)}\hat{T}_{l_2k_2}^{(S)}. \tag{21}$$

The product of two irreducible tensor operators can be expressed as a linear form on irreducible tensor operators [24],

$$\hat{T}_{l_1k_1}^{(S)}\hat{T}_{l_2k_2}^{(S)} = \sqrt{(2l_1+1)(2l_2+1)}\sum_{L,M}(-1)^{2S+L}C_{l_1k_1,l_2k_2}^{LM}\begin{Bmatrix} l_1 & l_2 & L \\ S & S & S\end{Bmatrix}\hat{T}_{LM}^{(S)}, \tag{22}$$

where $\begin{Bmatrix} l_1 & l_2 & L \\ S & S & S\end{Bmatrix}$ are $6j$-symbols. We use the following representation (which can be obtained by comparing equations 9.2.1 (5) and 8.2.1 (4) from [24]) of $6j$-symbols in terms of expansion on the Clebsch–Gordan coefficients,

$$\begin{Bmatrix} l_1 & l_2 & L \\ S & S & S\end{Bmatrix} = \frac{(-1)^{2S+l_1}}{\sqrt{(2l_1+1)}}\frac{F(l_2)F(l_1)}{F(L)}\sum_j a_j b_{jl_1}^{l_2} C_{l_2j,L0}^{l_1j}, \tag{23}$$

where

$$a_j = \frac{(-1)^j}{j!(2S+j+1)!}, \qquad b_{jl_1}^{l_2} = \left[\frac{(l_2+j)!(l_1+j)!}{(l_2-j)!(l_1-j)!}\right]^{1/2}, \tag{24}$$

and

$$F(L) = \sqrt{(2S+L+1)!(2S-L)!}. \tag{25}$$

Substituting (22) and (23) into (21) we get

$$\begin{aligned} \hat{f}\hat{g} = &\sum_{l_1,k_1}\sum_{l_2,k_2}\sum_{L,M}\sum_j (-1)^{-l_1} f_{l_1k_1}g_{l_2k_2}a_j b_{jl}^{L} \\ &\times\sqrt{\frac{(2l_1+1)(2L+1)}{2l_2+1}}\frac{F(l_2)F(l_1)}{F(L)}C_{LM,l_1-k_1}^{l_2k_2}C_{L0,l_1j}^{l_2j}\hat{T}_{LM}^{(S)}, \end{aligned} \tag{26}$$

where we have used the following transformation properties of the Clebsch–Gordan coefficients:

$$C_{l_1k_1,l_2k_2}^{LM} = (-1)^{l_1-k_1}\sqrt{\frac{2L+1}{2l_2+1}}C_{LM,l_1-k_1}^{l_2k_2}, \qquad C_{l_2j,L0}^{l_1j} = (-1)^{L}\sqrt{\frac{2l_1+1}{2l_2+1}}C_{L0,l_1j}^{l_2j}. \tag{27}$$

Using the integral representation [24] for a product of two Clebsch–Gordan coefficients in terms of the Wigner $D$-function,

$$C^{l_2 j}_{L0,l_1 j} C^{l_2 k_2}_{LM,l_1 - k_1} = \frac{2l_2+1}{8\pi^2} \int \mathrm{d}V\, D^L_{M0}(\phi,\theta,\psi) D^{l_1}_{-k_1 j}(\phi,\theta,\psi) D^{l_2 *}_{k_2 j}(\phi,\theta,\psi), \tag{28}$$

where $\mathrm{d}V = \mathrm{d}\phi\,\mathrm{d}\Omega$, and relations

$$\begin{aligned} D^L_{M0}(\phi,\theta,\psi) &= \sqrt{\frac{4\pi}{2L+1}} Y^*_{LM}(\theta,\phi), \\ D^{l_1}_{-k_1 j}(\phi,\theta,\psi) &= (-1)^{k_1} \sqrt{\frac{4\pi}{2l_1+1}} \sqrt{\frac{(l_1-j)!}{(l_1+j)!}} (S^+)^j Y_{l_1 k_1}(\theta,\phi), \\ D^{l_2 *}_{k_2 j}(\phi,\theta,\psi) &= \sqrt{\frac{4\pi}{2l_2+1}} \sqrt{\frac{(l_2-j)!}{(l_2+j)!}} (S^-)^j Y_{l_2 k_2}(\theta,\phi), \end{aligned} \tag{29}$$

where

$$S^{\pm} = \mathrm{i}\mathrm{e}^{\mp \mathrm{i}\psi} \left( \pm \cot\theta \frac{\partial}{\partial\psi} + \mathrm{i}\frac{\partial}{\partial\theta} \mp \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \right), \tag{30}$$

$$S^{\pm} D^L_{mm'}(\phi,\theta,\psi) = -\sqrt{L(L+1)} C^{Lm'\pm 1}_{Lm,1\pm 1} D^L_{mm'\pm 1}(\phi,\theta,\psi), \tag{31}$$

are contravariant components of the angular momentum operator in the rotating frame [24], we obtain from (26)

$$\begin{aligned} \hat{f}\hat{g} = \sum_{l_1,k_1} \sum_{l_2,k_2} \sum_{L,M} \sum_j f_{l_1 k_1} g_{l_2 k_2} \frac{a_j}{\sqrt{\pi}} \frac{F(l_2)F(l_1)}{F(L)} \hat{T}^{(S)}_{LM} \\ \times \int \mathrm{d}V\, Y^*_{LM}(\theta,\phi)(S^+)^j Y_{l_1 k_1}(\theta,\phi)(S^-)^j Y_{l_2 k_2}(\theta,\phi). \end{aligned} \tag{32}$$

Now we note that the function $F(L)$ (25) depends on the combination $L(L+1)$ rather than on $L$ itself and thus, we can write

$$F(L) D^L_{MM'}(\phi,\theta,\psi) = \tilde{F}(\mathcal{J}^2) D^L_{MM'}(\phi,\theta,\psi), \tag{33}$$

where $\tilde{F}(\mathcal{J}^2)$ is some function (whose explicit form is not needed for concrete calculations) of the Casimir operator $\mathcal{J}^2$,

$$\mathcal{J}^2 = -\left[ \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left( \frac{\partial^2}{\partial\phi^2} - 2\cos\theta\frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi^2} \right) \right], \tag{34}$$

$$[\mathcal{J}^2, S^{\pm}] = 0, \tag{35}$$

such that $\mathcal{J}^2 D^L_{MM'}(\phi,\theta,\psi) = L(L+1) D^L_{MM'}(\phi,\theta,\psi)$. Taking into account (20) and the relation

$$C^{SS}_{SS,L0} = \frac{\sqrt{(2S+1)!(2S)!}}{F(L)}, \tag{36}$$

we rewrite equation (32) in the following form:

$$\begin{aligned} \hat{f}\hat{g} = \frac{2N_S}{2S+1} \sum_j a_j \int \mathrm{d}V\, [\tilde{F}^{s-1}(\mathcal{J}^2)\hat{w}_{-s_1}(\theta,\phi)](S^+)^j \\ \times [\tilde{F}^{1-s_1}(\mathcal{J}^2) W^{(s_2)}_f(\theta,\phi)](S^-)^j [\tilde{F}^{1-s_2}(\mathcal{J}^2) W^{(s_3)}_g(\theta,\phi)], \end{aligned} \tag{37}$$

where

$$N_S = \sqrt{2S+1}[(2S+1)!(2S)!]^{(s_1+s_2-s)/2} = \sqrt{2S+1} F^{s_1+s_2-s}(0).$$

Integrating by parts in (37) we get

$$\hat{f}\hat{g} = \frac{2N_S}{2S+1}\sum_j a_j \int \mathrm{d}V\, \hat{w}_{-s}(\theta,\phi)\tilde{F}^{s-1}(\mathcal{J}^2)((S^+)^j \times [\tilde{F}^{1-s_1}(\mathcal{J}^2)W_f^{(s_1)}(\theta,\phi)](S^-)^j[\tilde{F}^{1-s_2}(\mathcal{J}^2)W_g^{(s_2)}(\theta,\phi)]). \quad (38)$$

Comparing (38) with (18) and (17) we obtain for the operator $\hat{L}_{fg}(\theta,\phi)$, defining the star product, the following expression:

$$\hat{L}_{fg}^{(s)}(\theta,\phi) = N_S\sum_j a_j \int \frac{\mathrm{d}\psi}{2\pi}\tilde{F}^{s-1}(\mathcal{J}^2)[((S^+)^j\tilde{F}^{1-s_1}(\mathcal{J}^2))_f \otimes ((S^-)^j\tilde{F}^{1-s_2}(\mathcal{J}^2))_g], \quad (39)$$

where the operators with subscript '$f$' act only on the $W_f^{(s_1)}(\theta,\phi)$, the operators with subscript '$g$' act only on the $W_g^{(s_2)}(\theta,\phi)$, whereas the external operator $\tilde{F}^{s-1}(\mathcal{J}^2)$ acts on the whole product. After integration over the angle $\psi$ we can rewrite equation (39) as

$$\hat{L}_{fg}^{(s)}(\theta,\phi) = N_S\sum_j a_j\tilde{F}^{s-1}(\mathcal{L}^2)[(S^{+(j)}\tilde{F}^{1-s_1}(\mathcal{L}^2))_f \otimes (S^{-(j)}\tilde{F}^{1-s_2}(\mathcal{L}^2))_g], \quad (40)$$

where $\mathcal{L}^2$ is the Casimir operator on the sphere

$$\mathcal{L}^2 = -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right], \qquad \mathcal{L}^2 Y_{L,M}(\theta,\phi) = L(L+1)Y_{L,M}(\theta,\phi), \quad (41)$$

such that

$$\tilde{F}(\mathcal{L}^2)Y_{L,M}(\theta,\phi) = F(L)Y_{L,M}(\theta,\phi), \quad (42)$$

and the function $F(L)$ is defined in (25). The symbolic powers $S^{\pm(j)}$ have been introduced in (40) according to

$$S^{\pm j} = \mathrm{e}^{\mp \mathrm{i}j\psi}S^{\pm(j)}, \quad (43)$$

such that

$$S^{\pm(j)} = \prod_{k=0}^{j-1}\left(k\cot\theta - \frac{\partial}{\partial\theta} \mp \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right). \quad (44)$$

It follows from (31) and (44) that

$$S^{\pm(j)}W_f^{(s)}(\theta,\phi) = 0, \qquad j > \deg\hat{f}. \quad (45)$$

The number of terms in the sum (40) is defined by the degree of non-linearity of the operators $\hat{f}$ and $\hat{g}$ (19), i.e. $j = 0, 1, \ldots, j_{\max}$; $j_{\max} = \min(\deg\hat{f}, \deg\hat{g})$. On the other hand, we can formally sum over '$j$' in equation (39) obtaining another representation for $\hat{L}_{fg}^{(s)}(\theta,\phi)$,

$$\hat{L}_{fg}^{(s)}(\theta,\phi) = N_S\int\frac{\mathrm{d}\psi}{2\pi}\tilde{F}^{s-1}(\mathcal{J}^2)\sigma(S_f^+\otimes S_g^-)(\tilde{F}^{1-s_1}(\mathcal{J}^2))_f \otimes (\tilde{F}^{1-s_2}(\mathcal{J}^2))_g \quad (46)$$

where the function $\sigma(z)$ is defined as

$$\sigma(z) = \sum_j \frac{(-1)^j}{j!(2S+j+1)!}z^j = \frac{1}{z^{S+1/2}}J_{2S+1}(2\sqrt{z}), \quad (47)$$

and $J_n(x)$ is the Bessel function.

Equations (40) and (46) are the main result of this paper. It is worth noting that although the operator function $\tilde{F}(\mathcal{L}^2)$ has explicitly entered into (40) and (46), the property (33) is sufficient to determine the action of $\hat{L}_{fg}(\theta,\phi)$ on the product of any pair of symbols $W_f^{(s_1)}$ and $W_g^{(s_2)}$.

One can observe from (40) and (46) that the star product acquires the simplest form for the Beresin $P$-symbol, $P_f(\theta,\phi) = W_f^{(s=1)}(\theta,\phi)$,

$$P_{fg}(\theta,\phi) = (2S+1)!\sum_j a_j (S^{+(j)} P_f(\theta,\phi))(S^{-(j)} P_g(\theta,\phi)),$$

where $a_j$ is defined in (24). For the $Q$-symbol, $Q_f(\theta,\phi) = W_f^{(s=-1)}(\theta,\phi)$ and the Stratonovich–Wigner $W$-symbol, $W_f(\theta,\phi) = W_f^{(s=0)}(\theta,\phi)$, we have correspondingly

$$Q_{fg}(\theta,\phi) = \frac{1}{(2S)!}\sum_j a_j \tilde{F}^{-2}(\mathcal{L}^2)(S^{+(j)}\tilde{F}^2(\mathcal{L}^2) Q_f(\theta,\phi))(S^{-(j)}\tilde{F}^2(\mathcal{L}^2) Q_g(\theta,\phi)),$$
$$W_{fg}(\theta,\phi) = \sqrt{2S+1}\sum_j a_j \tilde{F}^{-1}(\mathcal{L}^2)(S^{+(j)}\tilde{F}(\mathcal{L}^2) W_f(\theta,\phi))(S^{-(j)}\tilde{F}(\mathcal{L}^2) W_g(\theta,\phi)).$$

Also, various mixed relations are possible, for example, we can compute the Stratonovich–Wigner symbol $W_{fg}(\theta,\phi)$ starting from $Q$-symbols $Q_f(\theta,\phi)$ and $Q_g(\theta,\phi)$:

$$W_{fg}(\theta,\phi) = \frac{\sqrt{2S+1}}{(2S+1)!(2S)!}\sum_j a_j \tilde{F}^{-1}(\mathcal{L}^2)(S^{+(j)}\tilde{F}^2(\mathcal{L}^2) Q_f(\theta,\phi))(S^{-(j)}\tilde{F}^2(\mathcal{L}^2) Q_g(\theta,\phi)).$$

## 5. Example

To show how the expression (40) works we calculate the symbol for the operator $S_+S_-$, where $S_+$, $S_-$, $S_z$ are generators of the $2S+1$-dimensional representation of the $su(2)$ algebra ($S_+ \sim T_{11}$, $S_- \sim T_{1-1}$, $S_z \sim T_{10}$). First of all we note that

$$W_{S_z}^{(s)}(\theta,\phi) = \left(\frac{S}{S+1}\right)^{-s/2}\sqrt{S(S+1)}\cos\theta,$$
$$W_{S_\pm}^{(s)}(\theta,\phi) = \left(\frac{S}{S+1}\right)^{-s/2}\sqrt{S(S+1)}\sin\theta\,\mathrm{e}^{\pm \mathrm{i}\varphi}. \tag{48}$$

Due to the property (45) only two terms ($j = 0, 1$) contribute to the sum in (40). Taking into account that

$$W_{S_{z,\pm}}^{(s)}(\theta,\phi) \sim Y_{1m}(\theta,\phi), \qquad m = 0, \pm 1,$$

and (according to (42))

$$\tilde{F}(\mathcal{L}^2) Y_{1m}(\theta,\phi) = F(1) Y_{1m}(\theta,\phi),$$

we have

$$W_{S_+}^{(s_1)} *_s W_{S_-}^{(s_2)} = N_S F^{2-s_1-s_2}(1)\tilde{F}^{s-1}(\mathcal{L}^2)\left[\frac{1}{(2S+1)!}W_{S_+}W_{S_-} - \frac{1}{(2S+2)!}S_+^{(1)}W_{S_+}S_-^{(1)}W_{S_-}\right], \tag{49}$$

where

$$S^{\pm(1)} = -\frac{\partial}{\partial\theta} \mp \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}, \tag{50}$$

and we introduced the $s$-ordered star product $*_s$ according to

$$W_f^{(s_1)} *_s W_g^{(s_2)} = \hat{L}_{fg}^{(s)}(W_f^{(s_1)} W_g^{(s_2)}).$$

Substituting (48) into (49) we obtain after some algebra

$$W_{S_+}^{(s_1)} *_s W_{S_-}^{(s_2)} = \left(\frac{S}{S+1}\right)^{-s/2} \frac{F^{-s}(1)(2S+2)!}{4\sqrt{2S+1}} \tilde{F}^{s-1}(\mathcal{L}^2) \times \left[\frac{4S}{3} + 2\cos\theta - \left(\cos^2\theta - \frac{1}{3}\right)(2S+3)\right].$$

Because of

$$\cos^2\theta - \tfrac{1}{3} \sim Y_{20}(\theta,\phi), \qquad \cos\theta \sim Y_{10}(\theta,\phi),$$

we get

$$W_{S_+}^{(s_1)} *_s W_{S_-}^{(s_2)} = \left(\frac{S}{S+1}\right)^{-s/2} \frac{F^{-s}(1)(2S+2)!}{4\sqrt{2S+1}} \left[\frac{4S}{3} F^{s-1}(0) + 2F^{s-1}(1)\cos\theta - (2S+3)F^{s-1}(2)(\cos^2\theta - \tfrac{1}{3})\right], \tag{51}$$

where

$$F(2) = \sqrt{(2S-2)!(2S+3)!}, \qquad F(1) = \sqrt{(2S-1)!(2S+2)!},$$
$$F(0) = \sqrt{(2S)!(2S+1)!}.$$

Here we note that because of $s - 1 \leqslant 0$, the last term in equation (51) equals zero in the case of spin one-half, $S = 1/2$. Finally we obtain (for $S \geqslant 1$)

$$W_{S_+}^{(s_1)} *_s W_{S_-}^{(s_2)} = \frac{\sqrt{S(S+1)}}{2} \left(\frac{S}{S+1}\right)^{-s/2} \left[\frac{4S}{3}\left(\frac{S}{S+1}\right)^{(s-1)/2} + 2\cos\theta - \left(\cos^2\theta - \frac{1}{3}\right)(2S+3)\left(\frac{2S+3}{2S-1}\right)^{(s-1)/2}\right]. \tag{52}$$

In the case when $S = 1/2$, the last term in the above equation is absent. On the other hand, we have the equality

$$W_{S_+S_-}^{(s)} = W_{\mathcal{L}^2}^{(s)} - W_{S_z^2}^{(s)} + W_{S_z}^{(s)}. \tag{53}$$

Taking into account that

$$W_{\mathcal{L}^2}^{(s)} = S(S+1),$$
$$W_{S_z^2}^{(s)} = \frac{1}{2}(S(2S-1))^{(1-s)/2}((2S+3)(S+1))^{(1+s)/2}\left(\cos^2\theta - \frac{1}{3}\right) + \frac{S(S+1)}{3},$$

(note that in the case $S = 1/2$ we get $W_{S_z^2}^{(s)} = 1/4$) and substituting the above expressions into (53) we obtain (52), which proves the relation

$$W_{S_+}^{(s_1)} *_s W_{S_-}^{(s_2)} = W_{S_+S_-}^{(s)}.$$

## 6. Large spin limit

In the limit of large spin ($S \gg 1$), equations (40), (46) can be reduced to a rather simple form. First of all we note (see appendix A) that the functions $\tilde{F}(\mathcal{J}^2)$ (33) and $\sigma(z)$ (47) can be approximated as follows:

$$\tilde{F}(\mathcal{J}^2) \approx \frac{(2S+1)!}{\sqrt{2S+1}} \exp\left(\frac{\varepsilon}{2}\mathcal{J}^2\right),$$
$$\sigma(z) \approx \frac{1}{(2S+1)!} \exp(-\varepsilon z), \tag{54}$$

where

$$\varepsilon = \frac{1}{2S+1} \ll 1.$$

Then, the expression (46) for the operator $\hat{L}^{(s)}_{fg}(\theta,\phi)$ is approximated as

$$\hat{L}^{(s)}_{fg}(\theta,\phi) \approx \int \frac{\mathrm{d}\psi}{2\pi} \exp\left(\frac{\varepsilon(s-1)}{2}\mathcal{J}^2\right)\exp(-\varepsilon S^+_f \otimes S^-_g) \\ \times \exp\left(\frac{\varepsilon(1-s_1)}{2}\mathcal{J}^2_f \otimes 1_g + 1_f \otimes \frac{\varepsilon(1-s_2)}{2}\mathcal{J}^2_g\right). \tag{55}$$

Taking into account that the action of $\mathcal{J}^2$ on a product $AB$ is defined as

$$\mathcal{J}^2(AB) = [\mathcal{J}^2_A \otimes I_B + I_A \otimes \mathcal{J}^2_B - (S^+_A \otimes S^-_B + S^-_A \otimes S^+_B)](AB), \tag{56}$$

and using the Campbell–Baker–Hausdorff formula

$$\mathrm{e}^{\hat{f}}\mathrm{e}^{\hat{g}} = \mathrm{e}^{\hat{f}+\hat{g}+\frac{1}{2}[\hat{f},\hat{g}]+\cdots}, \tag{57}$$

we get (see appendix B)

$$\hat{L}^{(s)}_{fg}(\theta,\phi) \approx \int \frac{\mathrm{d}\psi}{2\pi} \exp\left[\frac{\varepsilon(s-s_1)}{2}\mathcal{J}^2_f \otimes I_g + \frac{\varepsilon(s-s_2)}{2} I_f \otimes \mathcal{J}^2_g \\ + \frac{\varepsilon}{2}((1-s)S^-_f \otimes S^+_g - (1+s)S^+_f \otimes S^-_g)\right]. \tag{58}$$

It is worth writing explicit expressions for some special cases. In the case when $s = s_1 = s_2$ equation (58) takes the form similar to that for the Heisenberg–Weyl group (3)

$$\hat{L}^{(s)}_{fg}(\theta,\phi) \approx \int \frac{\mathrm{d}\psi}{2\pi} \exp\left[-\frac{s\varepsilon}{2}(S^-_f \otimes S^+_g + S^+_f \otimes S^-_g) + \frac{\varepsilon}{2}(S^-_f \otimes S^+_g - S^+_f \otimes S^-_g)\right],$$

in particular, one obtains

$$P_{fg}(\theta,\phi) = \int \frac{\mathrm{d}\psi}{2\pi} \exp[-\varepsilon S^+_f \otimes S^-_g] P_f(\theta,\phi) P_g(\theta,\phi),$$
$$Q_{fg}(\theta,\phi) = \int \frac{\mathrm{d}\psi}{2\pi} \exp[\varepsilon(S^-_f \otimes S^+_g)] Q_f(\theta,\phi) Q_g(\theta,\phi),$$
$$W_{fg}(\theta,\phi) = \int \frac{\mathrm{d}\psi}{2\pi} \exp\left[\frac{\varepsilon}{2}(S^-_f \otimes S^+_g - S^+_f \otimes S^-_g)\right] W_f(\theta,\phi) W_g(\theta,\phi).$$

On the other hand, from (40) we can obtain another approximation for the operator $\hat{L}^{(s)}_{fg}(\theta,\phi)$ as a series on powers of $\varepsilon$,

$$\hat{L}^{(s)}_{fg}(\theta,\phi) = I_f \otimes I_g + \frac{\varepsilon}{2}[(s-s_1)\mathcal{L}^2_f \otimes I_g + (s-s_2) I_f \otimes \mathcal{L}^2_g \\ + ((1-s)S^{-(1)}_f \otimes S^{+(1)}_g - (1+s)S^{+(1)}_f \otimes S^{-(1)}_g)] + \mathrm{O}(\varepsilon^2), \tag{59}$$

where the operators $S^{\pm(1)}$ are defined in (50) and $\mathcal{L}^2$ is the Casimir operator on the sphere (41).

## 7. Conclusions

We have found an exact differential form for the star product for the family of $s$-parametrized $SU(2)$ Stratonovich–Weyl symbols. This result allows us to replace the operator algebra in the Hilbert space of spin-like systems by differential calculus in the corresponding (classical) phase space. The explicit form (40), (46) of the star product (taken at $s = s_1 = s_2$) allows us to write down the evolution equation (8) for the $s$-parametrized quasi-distribution functions,

$$\mathrm{i}\partial_t W_\rho^{(s)} = \hat{M}_{H\rho}^{(s)}(\theta,\phi)(W_H^{(s)} W_\rho^{(s)}), \tag{60}$$

where

$$\hat{M}_{H\rho}^{(s)}(\theta,\phi) = \hat{L}_{H\rho}^{(s)}(\theta,\phi) - \hat{L}_{\rho H}^{(s)}(\theta,\phi)$$

denote the Moyal brackets operator. Equation (60) is the quantum Liouville equation for quasi-distributions on the sphere.

In the limit case of large spin, we obtain from equation (59)

$$\hat{L}_{fg}^{(s)}(\theta,\phi) \approx I_f \otimes I_g + \frac{\varepsilon}{2}((1-s)S_f^{-(1)} \otimes S_g^{+(1)} - (1+s)S_f^{+(1)} \otimes S_g^{-(1)}),$$

which leads to the following approximate expression for the Moyal brackets operator:

$$\hat{M}_{H\rho}^{(s)}(\theta,\phi) \approx \varepsilon(S_H^{-(1)} \otimes S_\rho^{+(1)} - S_H^{+(1)} \otimes S_\rho^{-(1)}).$$

We have

$$S_f^{+(1)} \otimes S_g^{-(1)} - S_f^{-(1)} \otimes S_g^{+(1)} = \frac{2\mathrm{i}}{\sin\theta}\left(\frac{\partial}{\partial\phi_f} \otimes \frac{\partial}{\partial\theta_g} - \frac{\partial}{\partial\theta_f} \otimes \frac{\partial}{\partial\phi_g}\right) = 2\mathrm{i}\{,\}_P,$$

where $\{,\}_P$ denotes the Poisson brackets on the sphere, i.e. in the limit $S \gg 1$ the Moyal brackets are reduced to the Poisson brackets. Finally, we obtain that the approximate evolution equation for the $s$-parametrized quasi-distribution function $W_\rho^{(s)}$ takes the form

$$\partial_t W_\rho^{(s)} \approx 2\varepsilon\{W_\rho^{(s)}, W_H^{(s)}\}_P, \tag{61}$$

which is similar to the Heisenberg–Weyl case when the Moyal brackets reduce to the classical Poisson bracket (in the flat space) in the limit $\hbar \to 0$. Quantum corrections to the classical evolution equation (61) can be obtained by further expansion of the star product (40) in a series on the powers of $\varepsilon$. It is worth noting that the form of quantum corrections essentially depends on the type of $s$-ordering of the quasidistribution function.

The first-order partial differential equation (61) can be solved (for example, by the method of characteristics) and the evolution of the quasi-distribution function $W_\rho^{(s)}$ is given by

$$W_\rho^{(s)}(\theta_0,\phi_0|t) \approx W_\rho^{(s)}(\theta(\theta_0,\phi_0,t), \phi(\theta_0,\phi_0,t)|t=0),$$

where $\theta(\theta_0,\phi_0,t), \phi(\theta_0,\phi_0,t)$ are the classical trajectories on the sphere generated by the Hamiltonian $H$. Thus, in the quasiclassical limit of large spin, $S \gg 1$, the so-called 'truncated Wigner approach' [25], in which each point of the quantum probability distribution evolves along a classical trajectory, can be developed to give an approximate description of quantum dynamics of spin-like systems.

## Appendix A

We represent the function $F(L)$ (25) in the following manner:

$$F(L) = \frac{(2S+1)!}{\sqrt{2S+1}}\left[\prod_{k=0}^{L}\frac{1+\varepsilon k}{1-\varepsilon k}\right]^{\frac{1}{2}} = \frac{(2S+1)!}{\sqrt{2S+1}}\exp\left[\frac{1}{2}\sum_{k=0}^{L}\ln\frac{1+\varepsilon k}{1-\varepsilon k}\right].$$

Expanding the logarithm

$$\sum_{k=0}^{L}\ln\frac{1+\varepsilon k}{1-\varepsilon k} = \sum_{k=0}^{L}\sum_{n=0}^{\infty}\frac{2(\varepsilon k)^{2n+1}}{2n+1} = 2\sum_{k=0}^{L}\left[\varepsilon k + \frac{\varepsilon^3}{5}k^3 + \cdots\right]$$
$$= \varepsilon L(L+1) + \frac{\varepsilon^3}{10}[L(L+1)]^2 \ldots$$

we obtain equation (54).

We approximate the function $\sigma(z)$ as

$$\sigma(z)=\sum_{j=0}^{\infty}\frac{(-z)^j}{j!(2S+j+1)!}\approx\frac{1}{(2S+1)!}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{-z}{2S+1}\right)^j=\frac{1}{(2S+1)!}\exp\left(-\frac{z}{2S+1}\right).$$

## Appendix B

To multiply exponentials in (55) using the Campbell–Baker–Hausdorff formula (57),

$$\begin{aligned}\exp(\varepsilon\gamma\mathcal{J}^2)\exp(-\varepsilon S^+\otimes S^-)\approx{}&\exp[\varepsilon\gamma(\mathcal{J}^2\otimes I+I\otimes\mathcal{J}^2\\&-(S^+\otimes S^-+S^-\otimes S^+))-\varepsilon S^+\otimes S^-]\\&\times\exp\left[\frac{\varepsilon^2\gamma}{2}[S^+\otimes S^-,S^-\otimes S^+]+\mathrm{O}(\varepsilon^3)\right],\end{aligned}\tag{62}$$

with $\gamma=(s_1-1)/2$, we note that

$$[S^+\otimes S^-,S^-\otimes S^+]=S^+S^-\otimes S_z-S_z\otimes S^-S^++S_z\otimes S_z,\tag{63}$$

where

$$S_z=-\mathrm{i}\frac{\partial}{\partial\psi}.$$

But the symbols $W_f^{(s)}(\theta,\phi)$ do not depend on $\psi$, $S_zW_f^{(s)}(\theta,\phi)=0$, thus, the effect of action of the exponential in (62) is negligible in our approximation.

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